

Optimal Bread Slicing

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December 20, 2023

1 Introduction

While countless¹ branches of mathematics abstract real-world phenomena to explore richer understandings, the same has yet to be done for slicing bread. For 95 years, sliced bread has remained a phenomenon investigated empirically by bakers and food engineers. Expanding sliced bread to mathematics, we present a formal construction of bread slicing. From there, we offer straightforward utility functions to optimize various aspects of bread slicing. The vast majority of this work follows intuitively from real-world bread, and most proofs in this work are proof sketches, since full proofs would distract from the important contributions of this work.

2 Construction of Problem

Before we can understand how to optimally slice a loaf of bread, we must first construct a loaf of bread. For our purposes, we only care about the shape of the loaf. As such, we provide the following definitions.

Definition 2.1. (Loaf Unit). A set B is a loaf unit if and only if it can be expressed as:

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq w, g(x) \leq y \leq f(x)\}$$

where $w > 0$ and $f : [0, w] \rightarrow \mathbb{R}$, $g : [0, w] \rightarrow \mathbb{R}$ are bounded, Lebesgue-integrable functions such that for all $x \in [0, w]$, $g(x) \leq f(x)$. Note that this definition extends to sets that become loaf units through rigid motion.

Loaf units are a type of prism in three dimensions with infinite length. Their face is the space bounded by the functions f and g on the interval $[0, w]$. Loaf units are equivalent up to rotation, reflection, and translation in \mathbb{R}^3 , as their position is not relevant.

Definition 2.2. (Loaf). A set B is a loaf if and only if it can be expressed as the union of finitely-many, disjoint loaf units.

Our definition of a loaf allows for loaves with holes all the way through, or multiple, stacked overhangs. Without loss of generality, for the rest of this work, we consider a loaf made from only a single loaf unit, with no rotations, reflections, or translations.

Definition 2.3. (Slice). Given an angle $\theta \in (0, \frac{\pi}{2}]$ and a width $\delta > 0$, a (θ, δ, z_0) -slice is the set

$$S(\theta, \delta, z_0) = \{(x, y, z) \in \mathbb{R}^3 \mid z_0 + x \cot \theta \leq z \leq z_0 + x \cot \theta + \delta\}$$

Definition 2.4. (Adjacent Slices). Without loss of generality, we say two slices $S(\theta_1, \delta_1, z_{0,1})$ and $S(\theta_2, \delta_2, z_{0,2})$ are adjacent if and only if $\theta_1 = \theta_2$ and $z_{0,1} = z_{0,2} + \delta_2$.

Theorem 2.5. *Adjacent Slices Touch.* Given two adjacent slices, their intersection is the plane they share as a boundary.

¹a finite number

Proof. Take any two adjacent slices $S(\theta, \delta_1, z_0)$ and $S(\theta, \delta_2, z_0 + \delta_1)$. By definition, their intersection is the set of points $\{(x, y, z)\} \in \mathbb{R}^3$ such that

$$z_0 + x \cot \theta \leq z \leq z_0 + \delta_1 + x \cot \theta$$

and

$$z_0 + \delta_1 + x \cot \theta \leq z \leq z_0 + \delta_1 + x \cot \theta + \delta_2$$

This is only true for $z = z_0 + x \cot \theta + \delta_1$. As such, the intersection of the slices is the plane defined by that equation. \square

Definition 2.6. (Slice of Bread). A slice of bread T is the intersection of a loaf B and a slice $S(\theta, \delta, z_0)$.

Theorem 2.7. *Angle Independence.* The volume of a slice of bread is independent of the angle of the slice.

Proof. Take any slice of bread T_1 as the intersection of a loaf B and a slice $S_1 = S(\theta_1, \delta, z_1)$. Let T_2 be the slice of bread of the same loaf and slice $S_2 = S(\theta_2, \delta, z_2)$. By definition, we can write $T_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq w, g(x) \leq y \leq f(x), z_1 + x \cot \theta_1 \leq z \leq z_1 + x \cot \theta_1 + \delta\}$. The volume of the slice can be expressed as

$$\begin{aligned} m(T_1) &= m(S_1 \cap B) \\ &= \int_0^w \int_{g(x)}^{f(x)} \int_{z_1 + x \cot \theta_1}^{z_1 + x \cot \theta_1 + \delta} dz dy dx \\ &= \delta \int_0^w \int_{g(x)}^{f(x)} dy dx \\ &= \int_0^w \int_{g(x)}^{f(x)} \int_{z_2 + x \cot \theta_2}^{z_2 + x \cot \theta_2 + \delta} dz dy dx \\ &= m(S_2 \cap B) \\ &= m(T_2) \end{aligned}$$

As demonstrated, the volumes are equivalent. Therefore, the volume of a slice of bread is independent of the angle of the slice. \square

Corollary 2.8. *Position Independence.* The volume of a slice of bread is dependent only on the loaf and the width of the slice.

Some may argue that this construction of the bread slice means we may double-count the intersection of adjacent slices when calculating volume. This is of no consequence.

Theorem 2.9. *No Added Volume.* The volume of any finite number of adjacent slices is equivalent to the volume of a single slice that spans exactly those slices.

Proof. Consider any two adjacent slices $S_1 = S(\theta, \delta_1, z_0)$ and $S_2 = S(\theta, \delta_2, z_0 + \delta_1)$ and a loaf B . We want to show $m((S_1 \cap B) \cup (S_2 \cap B)) = m(S_1 \cap B) + m(S_2 \cap B)$. By theorem 2.5, the intersection of S_1 and S_2 is a plane, so $m(S_1 \cap S_2) = 0$. Thus, $m((S_1 \cap B) \cap (S_2 \cap B)) = m(S_1 \cap S_2 \cap B) = 0$. From here, we can consider the original equation.

$$\begin{aligned} m((S_1 \cap B) \cup (S_2 \cap B)) &= m(S_1 \cap B) + m(S_2 \cap B) - m((S_1 \cap B) \cap (S_2 \cap B)) \\ &= m(S_1 \cap B) + m(S_2 \cap B) \end{aligned}$$

\square

Informally, this theorem allows us to add back the faces of each slice with no meaningful change in the quantity of bread.

Definition 2.10. (Slice Face). The face of a slice of bread is the intersection of a loaf and one bounding plane of a slice. Given a slice $S = S(\theta, \delta, z_0)$, and a loaf B , the face of a slice can be expressed as $B \cap \{(x, y, z) \in S \mid z = z_0 + x \cot \theta\}$.

Theorem 2.11. *Area of Slice Face.* Given a slice of bread from the slice $S(\theta, \delta, z_0)$ and loaf B , the area of one face of the bread is as follows.

$$A_\theta = \csc \theta \int_0^w f(x) - g(x) dx$$

Proof. Take any slice $S = S(\theta, \delta, z_0)$ and loaf B . By definition, the slice is of the form $\{(x, y, z) \in S \mid 0 \leq x \leq w, g(x) \leq y \leq f(x), z = z_0 + x \cot \theta\}$. This is a region of the plane defined by the equation $z = z_0 + x \cot \theta$. Let this plane be denoted P . Note that the xz -region of the plane is defined as $\{(x, y, z) \in P \mid 0 \leq x \leq w, z = z_0 + x \cot \theta\}$. Call this region F . Let f_F and g_F on this region be the real-valued functions that give the bounds of y .

We can map this plane P onto the xy -plane by mapping the xy -plane onto P and subsequently taking the inverse. First, consider the shear $S : \mathbb{R}_{\{z=0\}}^3 \rightarrow \mathbb{R}^3$ such that $S(x, y, 0) \mapsto (x, y, x \cot \theta)$. Next, consider the rigid translation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $(x, y, z) \mapsto (x, y, z_0 + z)$. We can combine the x and z coordinates to a new coordinate direction on P in which $x \mapsto z_0 + \sqrt{x^2 + (x \cot \theta)^2} = z_0 + x \csc \theta$. Note the mappings are invertible, and $T \circ S$ is a C^1 -diffeomorphism from the xy -plane to P , so $S^{-1} \circ T^{-1}$ maps P to the plane. By change of variables,

$$\int_F f_F - g_F = \csc \theta \int_0^w f(x) - g(x) dx$$

Since the left-hand side calculates the face area, so does the right-hand side. □

Corollary 2.12. *Thickness of Slice.* Given a slice $S(\theta, \delta, z_0)$ and a loaf B , the thickness h of the slice of bread is $\delta \sin \theta$.

Proof. This follows from knowing the volume of the slice and the face area. Under the assumption that a slice of bread is an oblique prism, the volume is the product of the base and height. Solving for height results in $\delta \sin \theta$. □

Lemma 2.13. *Adjacent Slice Combined Thickness.* For a finite number of adjacent slices $S_1 = S(\theta, \delta_1, z_1), \dots, S_k = S(\theta, \delta_k, z_k)$, the thickness of a combined slice $S(\theta, \sum \delta_i, z_1)$ is the same as the sum of the thickness of each slice.

Given a fixed volume, we can vary θ , creating a trade-off between surface area and thickness. This next section will focus on optimizing this trade-off under various constraints.

3 Optimization

First, as a bit of notation, since A_θ represents the face area of a slice, let A be the area for a slice where $\theta = \frac{\pi}{2}$. This way, $A_\theta = A \csc \theta$.

Definition 3.1. Bread Optimization. Given two utility functions u_A and u_h , the utility of a slice $S = S(\theta, \delta, z_0)$ of a loaf B is $u(S, B) = u_A(A \csc \theta) + u_h(\delta \sin \theta)$.

Definition 3.2. Bruschetta Optimization. Given a loaf B , a bruschetta optimization is a bread optimization such that there exists a minimum thickness h_{min} such that for $h \geq h_{min}$, $u_h(h) = 1$, otherwise $u_h(h) = -\infty$. Also, $u_A(A \csc \theta) = cA \csc \theta$ for some constant $c > 0$.

Theorem 3.3. *Bruschetta Impossibility.* Given a fixed volume V , a loaf B , and a bruschetta optimization, there is a maximum number of slices that can be constructed such that $u_h \neq -\infty$.

Proof. Note that for a fixed volume of bread, the maximum thickness is $\max_{\theta} \delta \sin \theta = \delta$, so $\theta = \frac{\pi}{2}$. Given a large slice $S = S(\frac{\pi}{2}, l, z_0)$ of a loaf B , let $V < \infty$ be the total bread volume, and let h_{min} be the minimum bruschetta thickness. For the sake of clarity, we will call the large slice S a finite loaf. Note that $V = A \cdot l$.

Let $n = \lfloor \frac{l}{h_{min}} \rfloor$, so $l - h_{min} < nh_{min} \leq l$. By contradiction, assume you can slice the finite loaf S into some $n+k > n$ adjacent slices such that $u_h \not\equiv -\infty$. Let those adjacent slices be $S_1 = S(\frac{\pi}{2}, \delta_1, z_0), \dots, S_{n+k} = S(\frac{\pi}{2}, \delta_{n+k}, z_0 + \sum_{i=1}^{n+k-1} \delta_i)$. The combined thickness of the adjacent slices is $\sum_{i=1}^{n+k} \delta_i$. By lemma 2.13, since S is a slice representing the combined adjacent slices, $l \geq \sum_{i=1}^{n+k} \delta_i$. By the bruschetta optimization, for all slices S_i , $\delta_i \geq h_{min}$, so

$$\begin{aligned} \sum_{i=1}^{n+k} \delta_i &\geq \sum_{i=1}^{n+1} \delta_i \text{ WLOG} \\ &\geq \sum_{i=1}^{n+1} h_{min} \\ &= (n+1)h_{min} \\ &= nh_{min} + h_{min} \\ &> l - h_{min} + h_{min} \\ &= l \end{aligned}$$

which is a contradiction. Therefore, there exists a maximum number of slices that can be constructed such that $u_h \not\equiv -\infty$. In particular, that maximum is $n = \lfloor \frac{l}{h_{min}} \rfloor$. \square

Theorem 3.4. Optimal Bruschetta Slice. Given a fixed width of bread l and a number of slices n , if n does not exceed the Bruschetta Impossibility bound, then the bruschetta optimal angle is $\theta = \arcsin(\frac{nh_{min}}{l})$.

Proof. Clearly, bruschetta optimization requires maximum surface area given the h_{min} thickness constraint. Maximizing surface area with respect to θ requires minimizing θ under this constraint. For n slices of thickness h , since h evenly divides the thickness the bread n times, $h = \frac{l \sin \theta}{n} \implies \theta = \arcsin(\frac{nh}{l})$. Minimizing θ under the h_{min} constraint results in $h = h_{min}$, so optimally, $\theta = \arcsin(\frac{nh_{min}}{l})$. \square

Definition 3.5. Sandwich Optimization. Given a loaf B , a sandwich optimization is a bread optimization such that $u_A(A \csc \theta) = \frac{c_A}{A \csc \theta}$ and $u_h(\delta \sin \theta) = c_h \delta \sin \theta$ for some constants $c_h > 0, c_A > 0$.

Theorem 3.6. Optimal Sandwich. The sandwich optimal angle is always $\theta = \frac{\pi}{2}$.

Proof. Note that the slice angle must be in the interval $(0, \frac{\pi}{2}]$. Maximizing u_A with respect to θ requires maximizing θ . Furthermore, maximizing u_h with respect to θ requires maximizing θ . Therefore, the sandwich optimal angle is $\theta = \frac{\pi}{2}$. \square

4 Practical Applications

We will now consider a practical application of this optimization. In particular, we will derive a formula for the optimal cutting angle for a finite piece of bread.

Since most loaves of bread have finite length, but our construction uses infinite length, we can discuss a finite loaf as a slice $S(\frac{\pi}{2}, l, 0)$, where l is the length of the loaf. Let this loaf have a width of w . Assume you want to cut the bread into n slices with a minimum thickness h .

To reflect reality, cutting at a sharper angle will mean that you lose more bread on the ends. What you do with this extra bread is up to you, we only care about the n optimal slices. In particular, the length is reduced by $w \cot \theta$, so the new length is $l - w \cot \theta$. Using the thickness formula from before, we have the following equation for the slice thickness.

$$\begin{aligned}
h &= \frac{(l - w \cot \theta) \sin \theta}{n} \\
\implies nh &= l \sin \theta - w \cos \theta \\
\implies nh &= \sqrt{w^2 + l^2} \cdot \sin \left(\theta + \arctan \left(-\frac{w}{l} \right) \right) \\
\implies \theta &= \arcsin \left(\frac{nh}{\sqrt{w^2 + l^2}} \right) - \arctan \left(-\frac{w}{l} \right)
\end{aligned}$$

Thus, given a finite loaf of length l , width w , minimum thickness h , and a number of slices n , the optimal cutting angle can be directly calculated with the boxed formula. The authors have left an online calculator for this exact formula somewhere on the internet.

5 Conclusion

The conclusion is left as an exercise to the reader, preferably done with a loaf in hand.

6 Acknowledgements

Special thanks to Jonathan Ma. Without his help, we would have been one, sole contributor, in which case we would have to refer to ourselves as “I”. We acknowledge funding from the Jonik Multimodal Universe™.